The Limits of Decidable States on Open-Ended Evolution and Emergence*

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Abstract

Is undecidability a requirement for open-ended evolution (OEE)? Using algorithmic complexity theory methods, we propose robust computational definitions for open-ended evolution and adaptability of computable dynamical systems. Within this framework, we show that decidability imposes absolute limits to the growth of complexity on computable dynamical systems up to a logarithm of a logarithmic term. Conversely, systems that exhibit open-ended evolution must be undecidable, establishing undecidability as a requirement for such systems.

Complexity is assessed in terms of three measures: sophistication, coarse sophistication and busy beaver logical depth. These three complexity measures assign low complexity values to random (incompressible) objects. We conjecture that, for similar complexity measures that assign low complexity values, decidability imposes comparable limits to the stable growth of complexity and such behaviour is necessary for non-trivial evolutionary systems.

Finally, we show that undecidability of adapted states imposes novel and unpredictable behaviour on the individuals or population being modelled. Such behaviour is irreducible.

Theoretical Framework

A dynamical system is one that changes or evolves over time. Deterministic computable dynamical with discrete time are a class of systems where the initial state and, given the respective time, future states are known without gaps of information. Since their formalization by Church and Turing, the class of computable systems have shown unpredictability (Turing, 1936).

A deterministic discrete space evolutionary system is defined by an *evolution function* of the form $M_{t+1} = S(M_0, t, E)$, where M_0 is the *initial state* of the system, E is the *environment* and t is a natural number called the *time* variable of the system. The sequence of states $M_0, M_1, ..., M_t, ...$ is called the evolution of the system.

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For this class of systems we define open-ended evolution (OEE) as a process that has the property of producing families of objects of increasing *complexity*. This definition is based on two observations: open-ended evolutionary systems tend to produce families of objects of increasing *complexity* (Bedau, 1998; Auerbach and Bongard, 2014) and, for a number of complexity measures, the objects belonging to a fixed level of complexity are finite, implying that complexity increase over time is sufficient and necessary to keep producing new objects. In order to differentiate between trivial and interesting OEE evolutionary systems, we propose a stronger definition OEE called *strong OEE*, where we place restrictions to the drop on complexity that a system can display.

Our characterization of *adaptation* is based on a descriptive complexity inequality. Given a computable dynamical system that models an organism (or population), if an object is *adapted* then it must produce an acceptable approximation of its environment (Zenil et al., 2012) with a bounded information deficit, which is the amount of information that we need to describe the environment from a complete description of the object. Formally, it must meet the following inequality:

$$K(E|M) \leq \epsilon,$$

where M is the adapted object, E is the environment and $K(E|M)$ is the conditional algorithmic descriptive complexity (Kolmogorov, 1965; Chaitin, 1982) of E relative to M. Finally, we define a program such that $p(M) = E$ with $|p| \leq \epsilon$ as the behaviour of M.

Our characterization is robust given that the presented inequality bounds the amount of information needed to describe any computable method of obtaining E from M, be either a computable theory that describes adaptation or a computable model for an organism that tries to find solutions for E.

Finally, we say that an evolutionary system S *converges weakly* to E with degree ϵ for the initial state M_0 if there exist an infinity of times δ_i, called *adaptation times*, such that

$$K(E(M_0, \delta_i)|S(M_0, E, \delta_i)) \leq \epsilon.$$

Results

Weak convergence is decidable if there exist a computable function $\delta : i \mapsto \delta_i$ where δ_i are the adaptation times of the system. This is equivalent to the existence of an algorithm that decides if the system at the state M_{δ_i} is adapted.

The next theorem states that decidability imposes a tight limit to the growth of complexity of strong OEE evolutionary systems.

Theorem 1. *Let $S(M_0, E, t)$ be a weakly converging system with adaptation times $\delta_1, ..., \delta_i, ...$. If csoph, $depth_{bb}$ (Antunes and Fortnow, 2003) or $soph_c$ (Koppel, 1988), for a fixed c, show strong OEE that grows faster than $O(\log\log i)$ at an infinite subsequence, then the mapping $\delta : i \mapsto \delta_i$ is not computable.*

Furthermore, the following corollary shows the inexistente of partial solutions to decidability.

Corollary 2. *If $S(M_0, E, t)$ is a weakly converging system with adapted states $M_1, ..., M_i, ...$ that show strong OEE with speed greater than $O(\log\log i)$ at an infinite subsequence for the three stated complexity measures, then the mapping $\delta : i \mapsto \delta_i$ is not even semi-computable.*

The upper bound to the complexity that decidability imposes to OEE is extremely slow at double logarithm over the index of adapted states. If we disregard this increasingly insignificant growing rate, we can say that strong open-ended evolution implies undecidability of the adapted states.

Moreover, as the next theorem shows, undecidability implies that the behaviour of the adapted states must also be undecidable, establishing a path for emergent behaviour that cannot be predicted given a full description of the initial state and the behaviour of the system.

Theorem 3. *Let S be a non cyclical computable system with initial state M_0, E a dynamic environment and $\delta_1, ..., \delta_i, ...$ a sequence of times such that for each δ_i there exist a total function p_i such that $p_i(M_{\delta_i}) = E(\delta_i)$. If the function $p : i \mapsto p_i$ is computable, then the function $\delta : i \mapsto \delta_i$ is computable.*

The sequence of behaviours p_i is irreducible given that the sequence does not posses a shorter description than itself, otherwise it would be computable. These behaviours must also novel since they must be different enough from one to another in order to have irreducibility. We believe that these results can be extended to similar complexity measures that grows slower on random or incompressible objects. Formally:

Conjecture 4. Computability bounds the growing complexity rate to that of an order of the slowest growing infinite subsequence with respect to any *adequate* complexity measure C.

Conclusions

We have presented a formal and general mathematical model for adaptation within the framework of computable dynamical systems. This model exhibits universal properties for all computable dynamical systems, of which Turing machines are a subset.

Among other results, we have given formal definitions of open-ended evolution (OEE) and strong open-ended evolution. We also showed that decidability imposes universal limits to the growth of complexity of computable systems as measured by sophistication, coarse sophistication and busy beaver logical depth. Furthermore, as a direct implication theorem 1, undecidability of adapted states and unpredictability of the behaviour of the system at each state is a requirement for a system to exhibit strong open-ended evolution (up to a $O(\log\log t)$ term) with respect to the complexity measures known as sophistication, coarse sophistication and busy beaver logical depth, establishing a rigorous proof that undecidability and irreducibility of future behaviour is a requirement for the growth of complexity among the class of computable dynamical systems.

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